Finite action Yang - Mills solutions on the group manifold

1996 J. Phys. A: Math. Gen. 29 5001

(http://iopscience.iop.org/0305-4470/29/16/021)

J. Phys. A: Math. Gen. **29** (1996) 5001–5005. Printed in the UK

Finite action Yang–Mills solutions on the group manifold

T Dereli†‖, J Schray‡¶ and Robin W Tucker‡§

† Department of Physics, Middle East Technical University, 06531 Ankara, Turkey
‡ School of Physics and Chemistry, University of Lancaster, Lancaster LA1 4YB, UK

Received 13 November 1995, in final form 22 April 1996

Abstract. We demonstrate that the left (and right) invariant Maurer–Cartan forms for any semi-simple Lie group enable solutions of the Yang–Mills equations to be constructed on the group manifold equipped with the natural Cartan–Killing metric. For the unitary unimodular groups the Yang–Mills action integral is finite for such solutions. This is explicitly exhibited for the case of $SU(3)$.

1. Introduction

Classical solutions of the Yang–Mills field equations have proved to be of immense value in the development of both mathematical and physical theory. Instanton and monopole solutions have provided a fertile source of approximation techniques in quantum field theory and recent developments in duality theory have led to exciting new computational schemes in differential geometry based on supersymmetric Yang–Mills actions. Although self-duality has featured prominently in the construction of finite-action solutions not all such field configurations fall into this category. One of the earliest solutions found in $SU(2)$ Yang–Mills theory was the 'meron' [1–7]. Its generalizations gave rise to an interesting proposal for the confinement of quarks [8, 9]. Little attention has been devoted to the exploration of finite-action solutions associated with higher rank Lie groups in four [10] or higher dimensions [21].

It is the purpose of this note to point out that solutions of 'meron' type can occur in a much broader context. We demonstrate below that there exist similar finite-action solutions of the Yang–Mills field equations associated with *any compact semi-simple Lie group* if the action functional for the field equations is constructed over the group manifold itself in terms of a group invariant metric on the Lie algebra. Of particular interest for physics are the unitary unimodular groups. Such a broad class of solutions may have some relevance to one or more of the many compactification schemes that feature in the dimensional reduction of higher dimensional descriptions of the fundamental interactions. The explicit construction of any solution in this class depends upon the manner in which the group manifold is coordinated. We content ourselves here with the explicit construction of a finite-action solution associated with $SU(3)$.

‖ E-mail address: tdereli@rorqual.cc.metu.edu.tr
¶ E-mail address: J.Schray@lancaster.ac.uk
§ E-mail address: R.Tucker @lancaster.ac.uk

0305-4470/96/165001+05$19.50

2. Yang–Mills solutions

Suppose G is a semi-simple Lie group. Let $\{\omega^\alpha\}$ be the set of left invariant basis 1-forms on G that satisfy the Maurer–Cartan structure equations

$$\mathrm{d}\omega^\alpha = -\tfrac{1}{2} C_{\beta\gamma}{}^\alpha \omega^\beta \wedge \omega^\gamma \tag{1}$$

where $C_{\beta\gamma}{}^\alpha$ are the structure constants of the Lie algebra of G. If we define the dual basis of vectors $\{X_\alpha\}$ so that $\omega^\alpha(X_\beta) = \delta^\alpha{}_\beta$ then

$$[X_\alpha, X_\beta] = C_{\alpha\beta}{}^\gamma X_\gamma. \tag{2}$$

Let g denote the Cartan–Killing metric on the group manifold G so that

$$g_{\alpha\beta} \equiv g(X_\alpha, X_\beta) = C_{\alpha\gamma}{}^\delta C_{\beta\delta}{}^\gamma. \tag{3}$$

We also introduce an irreducible representation† of the Lie algebra of G by $N \times N$ matrices $\{\lambda_\alpha\}$ that satisfy

$$[\lambda_\alpha, \lambda_\beta] = C_{\alpha\beta}{}^\gamma \lambda_\gamma. \tag{4}$$

Thus the matrices are scaled in such a way that

$$\mathrm{tr}\,(\mathrm{ad}\,\lambda_\alpha\ \mathrm{ad}\,\lambda_\beta) = g_{\alpha\beta}. \tag{5}$$

Define a Yang–Mills potential 1-form

$$A = c\omega^\alpha \lambda_\alpha \tag{6}$$

where c is a constant to be determined. The corresponding gauge field 2-form is

$$F \equiv \mathrm{d}A + \tfrac{1}{2}[A, A] = -c(c-1)\,\mathrm{d}\omega^\alpha \lambda_\alpha. \tag{7}$$

Now consider the Hodge dual $*F$ with respect to the Cartan–Killing metric and substitute it in the Yang–Mills equation

$$d * F + A \wedge *F - *F \wedge A = 0. \tag{8}$$

We find

$$d * F = -c\,(c-1) d * \mathrm{d}\omega^\alpha \lambda_\alpha \tag{9}$$

and

$$A \wedge *F - *F \wedge A = c^2(c-1)\, C_{\beta\gamma}{}^\alpha \omega^\beta \wedge * \mathrm{d}\omega^\gamma \lambda_\alpha. \tag{10}$$

Excluding the trivial solutions $c = 0, 1$ the Yang–Mills equation is satisfied provided we have

$$d * \mathrm{d}\omega^\alpha = -c C_{\beta\gamma}{}^\alpha \omega^\beta \wedge * \mathrm{d}\omega^\gamma. \tag{11}$$

From equation (1) the left-hand side is

$$d * \mathrm{d}\omega^\alpha = -\tfrac{1}{2} C_{\beta\gamma}{}^\alpha d * (\omega^\beta \wedge \omega^\gamma) = -\tfrac{1}{2} C_{\beta\gamma}{}^\alpha d \left(\frac{1}{(N-2)!} \epsilon^{\beta\gamma}{}_{\delta\rho\ldots}\, \omega^\delta \wedge \omega^\rho \wedge \ldots \right) \tag{12}$$

where $\epsilon^{\beta\gamma}{}_{\delta\rho\ldots}$ is the Levi–Civita alternating density associated with the Cartan–Killing metric. With equations (1) and (3) it follows that

$$d * \mathrm{d}\omega^\alpha - \tfrac{1}{2} C_{\beta\gamma}{}^\alpha\, \mathrm{d}\omega^\delta \wedge *(\omega^\beta \wedge \omega^\gamma \wedge \omega_\delta) = \tfrac{1}{4} C_{\beta\gamma}{}^\alpha C_{\rho\sigma}{}^\delta \omega^\rho \wedge \omega^\sigma \wedge *(\omega^\beta \wedge \omega^\gamma \wedge \omega_\delta). \tag{13}$$

† For $G = SU(N)$ these matrices would be a set of anti-Hermitian $N \times N$ traceless matrices corresponding to the fundamental representation.

With the aid of the identities

$$\omega^\alpha \wedge *(\omega^\beta \wedge \omega^\gamma \wedge \omega^\delta) = g^{\alpha\beta} * (\omega^\gamma \wedge \omega^\delta) - g^{\alpha\gamma} * (\omega^\beta \wedge \omega^\delta) + g^{\alpha\delta} * (\omega^\beta \wedge \omega^\gamma) \tag{14}$$

$$\omega^\alpha \wedge *(\omega^\beta \wedge \omega^\gamma) = -g^{\alpha\beta} * \omega^\gamma + g^{\alpha\gamma} * \omega^\beta \tag{15}$$

this gives

$$d * \mathrm{d}\omega^\alpha = \tfrac{1}{2} g^{\beta\rho} g^{\gamma\sigma} C_{\beta\gamma}{}^{\alpha} C_{\rho\sigma}{}^{\delta} * \omega_\delta. \tag{16}$$

On the other hand the right-hand side of equation (11) may be written as

$$\begin{aligned} -cC_{\beta\gamma}{}^{\alpha}\omega^\beta \wedge * \mathrm{d}\omega^\gamma &= \tfrac{1}{2} c C_{\beta\gamma}{}^{\alpha}\omega^\beta \wedge C_{\rho\sigma}{}^{\gamma} * (\omega^\rho \wedge \omega^\sigma) \qquad \text{(from equation (1))} \\ &= -cC_{\beta\gamma}{}^{\alpha} C_{\rho\sigma}{}^{\gamma} g^{\beta\rho} * \omega^\sigma \qquad \text{(by equation (15))} \end{aligned} \tag{17}$$

with $g^{\alpha\beta}$ being the inverse of the Cartan–Killing metric. Hence the Yang–Mills equation is satisfied if

$$\tfrac{1}{2} C_{\beta\gamma}{}^{\alpha} C^{\beta\gamma\delta} = -cC_{\beta\gamma}{}^{\alpha} C^{\beta\delta\gamma} \tag{18}$$

where

$$C_{\alpha\beta\gamma} \equiv C_{\alpha\beta}{}^{\delta} g_{\delta\gamma} \tag{19}$$

are totally skew-symmetric. This implies $c = \frac{1}{2}$.

If we now restrict to the compact unimodular unitary groups, the group manifold has a finite volume $\int_G *1$ and the above solution yields a finite Yang–Mills action:

$$I[A] = \int_G \mathrm{tr}\,(F \wedge *F) = \frac{C_{\alpha\beta\gamma}C^{\alpha\beta\gamma}}{32} \int_G *1 \tag{20}$$

where we have used equation (15) and

$$\omega^\alpha \wedge *\omega^\beta = g^{\alpha\beta} * 1. \tag{21}$$

The construction of group volumes is not trivial in general [11–17] and the precise values depend on the choice of normalization of the group generators. We illustrate this by computing the above action for $G = SU(3)$.

If we denote the anti-Hermitian generators of the Lie algebra of $SU(3)$ by

$$\begin{aligned} &\lambda_1 = \frac{\sqrt{2}}{2}\begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \frac{\sqrt{2}}{2}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \frac{\sqrt{2}}{2}\begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &\lambda_4 = \frac{\sqrt{2}}{2}\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \frac{\sqrt{2}}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \frac{\sqrt{2}}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} \\ &\lambda_7 = \frac{\sqrt{2}}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad \lambda_8 = \frac{\sqrt{6}}{6}\begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{pmatrix} \end{aligned} \tag{22}$$

which are just $i/\sqrt{2}$ times the Gell–Mann 'λ-matrices' normalized such that $\mathrm{tr}\,(\lambda_i\lambda_j) = -\delta_{ij}$, then the group parametrization of $SU(3)$ given by Holland [17] reads

$$\mathcal{G}(\boldsymbol{\theta}) = \mathrm{e}^{-\sqrt{2}\theta^1\lambda_3}\mathrm{e}^{-\sqrt{2}\theta^2\lambda_2}\mathrm{e}^{-\sqrt{2}\theta^3\lambda_3}\mathrm{e}^{-\sqrt{6}\theta^4\lambda_8}\mathrm{e}^{-\sqrt{2}\theta^5\lambda_4}\mathrm{e}^{-\sqrt{2}\theta^6\lambda_3}\mathrm{e}^{-\sqrt{2}\theta^7\lambda_2}\mathrm{e}^{-\sqrt{2}\theta^8\lambda_3}. \tag{23}$$

The parameter ranges for the eight angles are $0 \leqslant \theta^1, \theta^3, \theta^6, \theta^8 \leqslant 2\pi$, $0 \leqslant \theta^4 \leqslant \pi$, and $0 \leqslant \theta^2, \theta^5, \theta^7 \leqslant \frac{1}{2}\pi$.

The Maurer–Cartan 1-forms $\omega^1, \omega^2, \ldots, \omega^8$ are defined by

$$\omega^i = -\,\mathrm{tr}\,(\lambda_i \mathcal{G}^{-1}\,\mathrm{d}\mathcal{G}). \tag{24}$$

The Cartan–Killing metric may be written in this parametrization as

$$-g_{CK} = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \cdots + \omega^8 \otimes \omega^8 \tag{25}$$

and the wedge product of the Cartan–Killing 1-forms, $*1$, is calculated to be

$$*1 = \omega^1 \wedge \omega^2 \wedge \ldots \wedge \omega^8 = 8\sqrt{3}\sin(2\theta^2)\sin(2\theta^5)\sin^2(\theta^5)\sin(2\theta^7)\,\mathrm{d}\theta^1 \wedge \mathrm{d}\theta^2 \wedge \ldots \wedge \mathrm{d}\theta^8. \tag{26}$$

We may integrate this to obtain the group volume $V_{SU(3)}$ of $SU(3)$ with respect to the Cartan–Killing metric volume element:

$$\begin{aligned} V_{SU(3)} = \int_{SU(3)} *1 &= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{\pi}\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{2\pi} 8\sqrt{3}\sin(2\theta^2)\sin(2\theta^5)\sin^2(\theta^5)\sin(2\theta^7) \\ &\quad \times \mathrm{d}\theta^1\,\mathrm{d}\theta^2\,\mathrm{d}\theta^3\,\mathrm{d}\theta^4\,\mathrm{d}\theta^5\,\mathrm{d}\theta^6\,\mathrm{d}\theta^7\,\mathrm{d}\theta^8 \\ &= 8\sqrt{3}\left[(2\pi)\times 1\times(2\pi)\times\pi\times\tfrac{1}{2}\times(2\pi)\times 1\times(2\pi)\right] = 64\sqrt{3}\pi^5. \end{aligned} \tag{27}$$

Equations (20) and (27) give explicit formulae for the Yang–Mills action associated with the 'meron' type solution discussed above. For $SU(3)$, this solution has both Pontryagin numbers zero.

3. Conclusion

We have shown that for any semisimple Lie group with general element $\mathcal{G}$ the potential form $A = c\mathcal{G}^{-1}\,\mathrm{d}\mathcal{G}$ is a solution of the Yang–Mills equation on a group manifold with the natural Cartan–Killing metric, if c takes the value $\frac{1}{2}$. By a simple local gauge transformation $A = (1-c)\,\mathcal{G}\,\mathrm{d}\mathcal{G}^{-1}$ is an equivalent solution. If the group is unimodular, both left and right invariant group volume elements can be chosen to coincide, so for compact groups the Yang–Mills action is finite for such solutions. For $SU(3)$ the result is

$$|I[A]| = \mu^2 96\sqrt{3}\pi^5$$

for the metric $g = \mu g_{CK}$ where μ is any real constant.

We have also briefly examined the more general ansatz

$$A = \sum_{i=1}^{8} c_i \omega^i \lambda_i.$$

For a constant vector $\{c_i\}$, $\sum_{i=1}^{8} c_i\omega^i$ is the most general left invariant 1-form on the group. One finds that A generates a solution provided the components $\{c_i\}$ satisfy a set of coupled quartic equations. For $SU(3)$ we demand that these equations have real solutions. Such solutions exist but may be shown to be related to the solution discussed above by constant gauge transformations (automorphisms of the Lie algebra). The remaining complex solutions give rise to finite but complex Yang–Mills actions.

There have been a number of papers [18–20] devoted to the generalizations of the Yang–Mills solutions to higher dimensions. In their search for the higher-dimensional analogues of instantons and merons these papers have modified the form of the original Yang–Mills action in an attempt to preserve the conformal properties that the theory exhibits in four dimensions. In this note we have given a simple proof of the existence of finite action solutions to the original Yang–Mills equation expressed in terms of the natural group invariant metric provided by the group manifold itself.

Acknowledgments

RWT and TD are grateful to NATO grant number CRG-941210 which made this work possible. RWT is also grateful to the Human Capital and Mobility Programme of the European Union for partial support. TD would like to thank M S Marinov for useful discussions.

References

[1] Alfaro V De, Fubini S and Furlan G 1976 A new classical solution of the Yang–Mills field equations *Phys. Lett.* **65B** 163
[2] Alfaro V De, Fubini S and Furlan G 1977 Properties of $O(4) \times O(2)$ symmetric solutions of the Yang–Mills field equations *Phys. Lett.* **72B** 203
[3] Alfaro V De, Fubini S and Furlan G 1978 Classical solutions of generally invariant gauge theories *Phys. Lett.* **73B** 463
[4] Glimm J and Jaffe A 1978 Multiple meron solutions of the classical Yang–Mills equations *Phys. Lett.* **73B** 167
[5] Cervero J, Jacobs L and Nohl C R 1977 Elliptic solutions of classical Yang–Mills theory *Phys. Lett.* **69B** 351
[6] Howe P S and Tucker R W 1978 An approach to $SU(2)$ gauge fields in Minkowski space-time *Nucl. Phys.* B **138** 73
[7] Minzoni A A, Muciño J and Rosenbaum M 1994 On the structure of Yang–Mills fields in compactified Minkowski space *J. Math. Phys.* **35** 5642
[8] Callan C, Dashen R and Gross D J 1977 A mechanism for quark confinement *Phys. Lett.* **66B** 375
[9] Glimm J and Jaffe A 1979 Charges, vortices and confinement *Nucl. Phys.* B **149** 49
[10] Chakrabarti A and Comtet A 1979 Merons and meronlike configurations *Phys. Rev.* D **19** 3050
[11] Marinov M S 1980 Invariant volumes of compact groups *J. Phys. A: Math. Gen.* **13** 3357–66
[12] Murnaghan D 1962 *The Unitary and Rotation Groups* (Washington, DC: Spartan Books)
[13] Beg M and Ruegg H 1965 A set of harmonic functions for the group $SU(3)$ *J. Math. Phys.* **6** 677
[14] Chacon E and Moshinsky M 1966 Representations of finite U_3 transformations *Phys. Lett.* **23** 567
[15] Nelson T J 1967 A set of harmonic functions for the group $SU(3)$ as specialized matrix elements of a general finite transformation *J. Math. Phys.* **8** 857
[16] Macfarlane A J, Sudbery A and Weisz P H 1968 On Gell–Mann's λ-matrices, d- and f-tensors, octets, and parametrizations of $SU(3)$ *Commun. Math. Phys.* **11** 77
[17] Holland D F 1969 Finite transformations of *SU3 J. Math. Phys.* **10** 531
[18] Tchrakian D H 1985 Spherically symmetric gauge field configurations with finite action in 4p-dimensions (p=integer) *Phys. Lett.* **150B** 360
[19] Saçlioğlu C 1986 Scale invariance and self-duality in higher dimensions *Nucl. Phys.* B **277** 487
[20] O'Brien G M and Tchrakian D H 1987 Meron field configurations in every even dimension *Phys. Lett.* **194B** 411
[21] Popov A D 1992 Meron-type solutions of the Yang–Mills equations in 4n dimensions *Europhys. Lett.* **19** 465